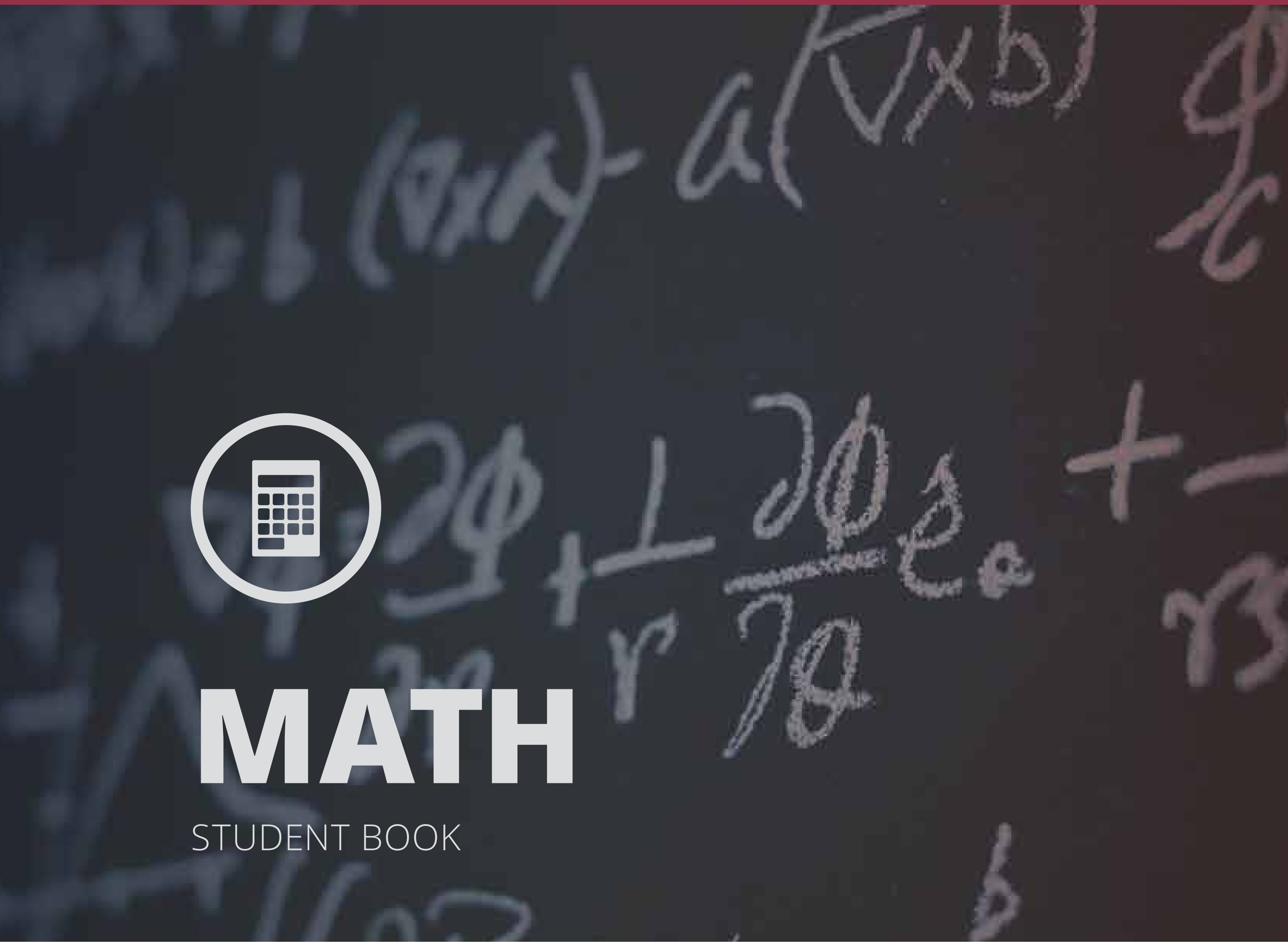
LIFEPAC®



MATH

STUDENT BOOK

10th Grade | Unit 2

Alpha Omega
PUBLICATIONS

MATH 1002

Proof

LIFEPAC Test is located in the center of the booklet. Please remove before starting the unit.

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Proof

Introduction

One of the main categories of items of our geometric system is the properties we call theorems. Theorems are general statements that can be proved. This LIFEPAC® presents methods of proving theorems by using logical thinking and deductive reasoning.

Objectives

Read these objectives. The objectives tell you what you will be able to do when you have successfully completed this LIFEPAC. When you have finished this LIFEPAC, you should be able to:

1. Identify the various compound sentences.
2. Use truth tables for compound sentences.
3. Explain the difference between inductive and deductive reasoning.
4. Use deductive reasoning in proofs.
5. Describe the six parts of a two-column proof.
6. Write an indirect proof.

Survey the LIFEPAC. Ask yourself some questions about this study and write your questions here.

1. LOGIC

Section Objectives

Review these objectives. When you have completed this section, you should be able to:

1. Identify various compound sentences.
2. Use truth for compound sentences.

A statement in mathematics has a narrow technical meaning. In LIFEPAC 1001 we learned that we must carefully define our terms in mathematics to avoid any misunderstandings. Accordingly, we make the following definition of a statement.

> **DEFINITION**
>
> **Statement:** A sentence that is either true or false, but not both.

Some of the following sentences are statements:

1. 3 times 5 equals 15.
2. $x + 8 = 20$.
3. A ray is a segment.
4. He is the president.
5. The lights are on.
6. Points *A* and *B* determine a line.

We can determine if they are true or if they are false.

Sentence 1 is a statement. From arithmetic facts we know the statement is true.

Sentence 3 is a statement. We can compare the definition of a ray and a segment and determine that sentence 3 is false.

Sentence 5 is a statement. We can see the light is on or touch the bulb to see if it is hot and then determine if sentence 5 is true or false. Sentence 6 is a true statement. One of our postulates supports its truth.

We cannot determine whether if sentences 2 and 4 are true or not until we have more information: what number does *x* represent, and who is he. So sentences 2 and 4 are not statements.

Write *statement* before each sentence that fits the definition of a statement.

1.1 ______________ The sun comes up in the east.

1.2 ______________ 5 x 6 = 35.

1.3 ______________ The moon is made of green cheese.

1.4 ______________ Water boils at 212˚ F.

1.5 ______________ A line has two end points.

1.6 ______________ Open the door.

1.7 ______________ $x^2 + 3x + 12 = 0$.

1.8 ______________ He is five years old.

1.9 ______________ Genesis is the last book of the Bible.

1.10 ______________ *AB* is the name of a line.

CONJUNCTION

Besides the simple statements you have just looked at, we can have more complicated ones, like these. Note that each statement is either true or false.

a) I cannot go with you because I have nothing to wear, my feet hurt, and I broke my glasses.
b) Gloria is a seamstress and Diane is a singer, while Gladys is neither.
c) Either I'll go to the game Friday or I'll go to the party.
d) Two lines intersect, are parallel, or are skew.
e) If two lines intersect, then the vertical angles formed are equal.

In some of these examples the full statement is made up of simpler statements. We can combine two simple statements to give a more complicated statement. For example:

Mr. Dale lives on a hill.
Mr. Hill lives in the dale.

We can combine these two statements with the word *and*:

Mr. Dale lives on a hill *and* Mr. Hill lives in the dale.

Here are some more examples of combining two statements with and:

a) The sun is shining and today is Monday.
b) I'm going to the game and to the party.
c) 5 + 3 = 8 and 6 • 3 = 20.
d) A triangle has three sides and water is wet.
e) A theorem is a proved statement and a postulate is accepted without proof.

All of these statements can be put into the pattern, *p* and *q*, where *p* represents the first statement and *q* represents the second statement. The new statement *p* and *q* is called the conjunction of *p* and *q*.

DEFINITION

Conjunction: A statement formed by combining two statements with the word *and*.

Since a conjunction is a statement, we want to be able to tell whether it is true or false. The following statements are all conjunctions. See if you can decide which ones are true and which ones are false. Then try to find a general rule for finding the truth of a conjunction.

a) 2 • 2 = 4 and 2 + 2 = 4
b) 3 + 2 = 5 and 3 • 2 = 7
c) 5 + 1 = 7 and 5 • 1 = 5
d) 6 – 2 = 8 and 12 ÷ 6 = 72

You probably had no trouble coming to the same conclusion that mathematicians have agreed upon for finding the truth of conjunctions. That is, *p and q* is true only when both *p and q* are true. If one of the parts is false, then the whole statement is false. We can put this rule into a truth table:

CONJUNCTION		
p	***q***	***p & q***
T	T	T
T	F	F
F	T	F
F	F	F

Note: A truth table is a list of all possible combinations of T and F for the statements involved in the statement.

The truth table tells us that *p and q* is true (T) only when both *p and q* are true, (top row in table). It is false (F) in all other cases.

Write true or false to tell whether the following conjunctions are true or false. Use the truth table if you need it.

1.11 ____________ 6 + 3 = 9 and 4 • 4 = 20.

1.12 ____________ Dogs have four legs and cats have ten lives.

1.13 ____________ January has 31 days and May has 31 days.

1.14 ____________ Sugar is sour and lemons are sweet.

1.15 ____________ A triangle has four sides and a rectangle has three sides.

1.16 ____________ A plane has at least three noncollinear points and a line has at least two points.

1.17 ____________ A square root of 16 is 4 and a square root of 9 is -3.

1.18 ____________ The intersection of two planes is a point and two lines intersect in a point.

1.19 ____________ The alphabet has 28 letters and a week has seven days.

1.20 ____________ 6 + 3 = 9 and 4 • 4 = 16.

DISJUNCTION

We can combine two statements in another way by using the word *or*. For example:

a)	3 + 2 = 5	*or*	7 • 7 = 49
b)	12 – 5 = 7	*or*	5 – 5 = 10
c)	5 • 3 = 20	*or*	7 > -2
d)	16 • 3 = 163	*or*	-6 < -10

These statements are of the form *p or q* where *p* represents the first statement and *q* the second statement. The new statement *p or q* is called the *disjunction of p and q*.

DEFINITION

Disjunction: A statement formed by combining two statements with the word *or*.

The following truth table illustrates disjunction.

DISJUNCTION		
p	***q***	***p* or *q***
T	T	T
T	F	T
F	T	T
F	F	F

Notice that a disjunction is true when either *one* or *both* of its parts are true. It is false only when both parts are false.

Write true or false to tell whether the following disjunctions are true or false. Use the truth table if you need it.

1.21 ____________ January is the first month of the year or December is the last month.

1.22 ____________ $5 \cdot 3 = 15$ or $7 + 5 = 20$.

1.23 ____________ A line has only one point or a plane has at least 3 points.

1.24 ____________ George Washington was President of Mexico or Abe Lincoln was President of Spain.

1.25 ____________ 5 is less than 7 or 7 is less than 12.

1.26 ____________ A segment has a midpoint or $\overrightarrow{AB}$ is the name of a ray.

1.27 ____________ A line and a point outside the line are in exactly one plane or two planes intersect in a plane.

1.28 ____________ Some roses are red or some violets are blue.

1.29 ____________ *Collinear* means points on the same plane or coplanar means points on the same line.

1.30 ____________ In this diagram, $RS + ST = SR$ or $ST + RS = RT$.

NEGATION

Where *p* represents a statement, a new statement, *p is false* can be formed. This statement is usually shortened to *not p* and is written ~*p*. The new statement formed is called the *negation of p* or the negative of *p*.

DEFINITION

Negation: If *p* is a statement, the new statement, *not p* or *p is false*, is called the negation of *p*

Some examples of negations are:

a) *p*: It is raining.
not *p*: It is not raining.

b) *p:* $6 + 3 = 12$
~*p:* $6 + 3 \neq 12$

c) *p:* The lights are on.
not *p:* The lights are not on; or the lights are off.

d) *p:* A triangle has six sides.
~*p:* A triangle does not have six sides.

e) *p:* It is false that spinach is good for you.
~*p:* Spinach is good for you.

When we form the negation of a statement in words, we do not usually place not in front of the original statement. Logically, to do so would be correct; but the resulting statement might sound strange. Instead we place the not in the sentence where it sounds better.

The truth table for negation is quite simple:

NEGATION	
p	***~p***
T	F
F	T

If a statement is true, its negation is false. If a statement is false, its negation is true.

Write the negation of the following statements.

1.31	______________________________	The grass is green.
1.32	______________________________	This rose is white.
1.33	______________________________	$5 + 4 = 90$
1.34	______________________________	$5 > -5$
1.35	______________________________	Geometry is interesting.
1.36	______________________________	A line has no length.
1.37	______________________________	All pigs are fat.
1.38	______________________________	My dog has fleas.
1.39	______________________________	Two points determine a line.
1.40	______________________________	A line does not have a midpoint.

CONDITIONAL

Where *p* and *q* represent statements, the compound statement written *if p, then q* is called a conditional or implication and is expressed in symbols by $p \rightarrow q$.

DEFINITION

Conditional or Implication: A statement formed from two given statements by connecting them in the form of *if* __________, *then* __________.

Hypothesis: the *if* clause in a statement.

Conclusion: the *then* clause in a statement.

Statement p is called the *hypothesis*. Statement q is called the *conclusion*. The truth table for a conditional shows the conditions under which this new statement is true.

CONDITIONAL		
p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Notice that $p \to q$ is always true except when p, the hypothesis, is true and q, the conclusion, is false.

The example that follows will help you to understand the rationale of the truth in the table.

Model: Roger makes this promise. "If I get a job, then I'll buy a motorcycle." Four possibilities are:

1) Roger gets a job. (p is true)
He buys a motorcycle. (q is true)
He keeps his promise. ($p \to q$ is true)

2) Roger gets a job. (p is true)
He does not buy the motorcycle. (q is false)
He broke his promise. ($p \to q$ is false)

3) Roger does not get a job. (p is false)
He still buys the motorcycle. (q is true)
He has not broken his promise. ($p \to q$ is true)

4) Roger does not get a job. (p is false)
He does not buy the motorcycle. (q is false)
He has not broken his promise. ($p \to q$ is true)

Write true or false to tell whether the conditional $p \to q$ is true or false. Use the truth table if needed.

1.41 ____________ If 3 + 2 = 5, then 5 + 5 = 10.

1.42 ____________ If 6 • 3 = 18, then 4 + 8 = 20.

1.43 ____________ If 6 > 10, then 8 • 3 = 24.

1.44 ____________ If 3 • 2 = 5, then 6 < 0.

1.45 ____________ If dogs have five legs, then cats have three tails.

1.46 ____________ If ice is hot, then rain is wet.

1.47 ____________ If a week has seven days, then a year has twenty months.

1.48 ____________ If water is wet, then 5 + 3 = 15.

1.49 ____________ If A and B are the names of two points, then $\overleftrightarrow{AB}$ is the name of a line through A and B.

1.50 ____________ If 5 = 3 + 2, then 3 + 2 = 5.

CONVERSE, INVERSE, CONTRAPOSITIVE

We can do three things to a conditional statement to get new statements. We can interchange the *p* with the *q* and get a new conditional, *if q*, *then p*. The hypothesis becomes the conclusion and the conclusion becomes the hypothesis. This new conditional is called the *converse* of the original conditional.

> **DEFINITION**
>
> **Converse of a conditional:** A statement formed by interchanging the hypothesis and the conclusion in a conditional statement.

Some examples of conditionals and their converses are:

a) **Conditional:** If today is Monday, then tomorrow is Tuesday. (True)

Converse: If tomorrow is Tuesday, then today is Monday. (True)

b) **Conditional:** If 5 + 5 = 20, then 3 + 2 = 5. (True)

Converse: If 3 + 2 = 5, then 5 + 5 = 20. (False)

c) **Conditional:** If a point lies on a line, then the line contains the point. (True)

Converse: If a line contains a point, then the point lies on the line. (True)

d) **Conditional:** If 3 > 5, then 3 ≠ 4. (True)

Converse: If 3 ≠ 4, then 3 > 5. (False)

A conditional and its converse are different statements. The examples given show that the converse of a true conditional may be true or it may be false.

When the conditional, $p \rightarrow q$, is true and the converse, $q \rightarrow p$, is also true, then the statements, *p* and *q*, are said to be *equivalent* statements and can be written $p \leftrightarrow q$. Equivalent statements have the same truth value. Either both are true statements or both are false statements.

In example a), saying "today is Monday," is equivalent to saying "tomorrow is Tuesday."

In example c), "a point lies on the line," is equivalent to "the line contains the point."

Where $p \rightarrow q$ is a conditional, a new conditional can be formed by negating both the hypothesis and the conclusion: $\sim p \rightarrow \sim q$. The new conditional is called the *inverse* of the conditional.

> **DEFINITION**
>
> **Inverse of a conditional:** A new statement formed by negating both the hypothesis and the conclusion.

Some examples of conditionals and their inverses are:

a) **Conditional:** If 5 + 5 = 20, then 3 + 2 = 5. (True)

Inverse: If 5 + 5 ≠ 20, then 3 + 2 ≠ 5. (False)

b) **Conditional:** If today is Friday, then yesterday was Thursday. (True)

Inverse: If today is not Friday, then yesterday was not Thursday. (True)

c) **Conditional:** If a point lies in a plane, then the plane contains the point. (True)

Inverse: If a point does not lie in a plane, then the plane does not contain the point. (True)

A conditional and its inverse are different statements. The examples show that the inverse of a true conditional may be true but it may also be false. The conditional and its inverse are not equivalent statements.

When $p \rightarrow q$ is a conditional, a new conditional can be formed by interchanging the hypothesis and conclusion and negating both of them. $\sim q \rightarrow \sim p$ is the new conditional formed. It is called the *contrapositive* of the conditional.

> **DEFINITION**
>
> **Contrapositive of a conditional:** A new statement formed by exchanging the hypothesis and the conclusion and negating both of them.

Some examples of conditionals and their contrapositives are:

a) **Conditional:** If today is Sunday, then tomorrow is Monday. (True)

Contrapositive: If tomorrow is not Monday, then today is not Sunday. (True)

b) **Conditional:** If a triangle has a right angle, then it is a right triangle. (True)

Contrapositive: If a triangle is not a right triangle, then it does not have a right angle. (True)

c) **Conditional:** If $5 + 8 \neq 13$, then $3 \cdot 2 = 6$. (True)

Contrapositive: If $3 \cdot 2 \neq 6$, then $5 + 8 = 13$. (True)

d) **Conditional:** If $6 > 3$, then $2 > 5$. (False)

Contrapositive: If $2 \ngtr 5$, then $6 \nless 3$. (False)

Notice in the examples that when the conditional is true, its contrapositive is true. When the conditional is false, its contrapositive is also false. Thus a conditional and its contrapositive are equivalent statements and we can write:

$$(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$$

COMPILED TRUTH TABLE

STATEMENTS		NEGATION		CONDITIONAL	CONTRAPOSITIVE	CONVERSE	INVERSE
p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$\sim q \rightarrow \sim p$	$q \rightarrow p$	$\sim p \rightarrow \sim q$
T	T	F	F	T	T	T	T
T	F	F	T	F	F	T	T
F	T	T	F	T	T	F	F
F	F	T	T	T	T	T	T

Notice the arrangement of the T's and F's in the *conditional* and the *contrapositive* columns. They are the same. Also notice the T's and F's in the *converse* and *inverse* columns. They are the same. This arrangement reveals that a conditional and its contrapositive are equivalent statements. Also, the converse and inverse of a conditional are equivalent statements.

Perform these two-step problems. If "*p*" denotes the statement "$x > 7$" and "*q*" the statement "$x > 5$", a. translate the given conditional into an if – – then statement about *x* (choose one value for "*x*" that satisfies both "*p*" and "*q*", use that number to work all these problems), and b. tell whether your statement is true or false.

True or False

1.51 $p \to q$ a. ______________________ b. __________

1.52 $p \to \sim q$ a. ______________________ b. __________

1.53 $\sim p \to q$ a. ______________________ b. __________

1.54 $\sim p \to \sim q$ a. ______________________ b. __________

1.55 $q \to p$ a. ______________________ b. __________

1.56 $q \to \sim p$ a. ______________________ b. __________

1.57 $\sim q \to p$ a. ______________________ b. __________

1.58 $\sim q \to \sim p$ a. ______________________ b. __________

Write the converse, inverse, and contrapositive of each of the following statements.

1.59 If two angles are adjacent, then the two angles have the same vertex.

Converse: ______________________

____

Inverse: ______________________

____

Contrapositive: ______________________

____

1.60 If today is Thursday, then tomorrow is Wednesday.

Converse: ______________________

Inverse: ______________________

Contrapositive: ______________________

1.61 If a polygon is a square, then it is a rectangle.

Converse: ______________________

Inverse: ______________________

Contrapositive: ______________________

1.62 If two lines intersect, then their intersection is one point.

Converse: ______________________________

____

Inverse: ______________________________

____

Contrapositive: ______________________________

____

1.63 Write a conditional statement and its converse such that the conditional is true but the converse is false.

Conditional: ______________________________

____

Converse: ______________________________

____

1.64 Write a conditional statement and its converse such that the conditional is true and the converse is true.

Conditional: ______________________________

____

Converse: ______________________________

____

1.65 Write a conditional statement and its contrapositive such that the conditional is true but the contrapositive is false.

Conditional: ______________________________

____

Converse: ______________________________

____

TEACHER CHECK ____________ initials ____________ date

Review the material in this section in preparation for the Self Test. The Self Test will check your mastery of this particular section. The items missed on this Self Test will indicate specific areas where restudy is needed for mastery.

SELF TEST 1

Write the answer in the blank to make true statements (each answer, 3 points).

1.01 When two statements are connected with the word *and*, the new statement is called a ______________________________ .

1.02 When two statements are connected with the word *or*, the new statement is called a ______________________________ .

1.03 Two statements connected with the words *if...then* form a new statement called a a. ____________________________ or b. ____________________________________ .

1.04 If a statement is true, its negation is ____________________________________ .

1.05 The converse of $p \rightarrow q$ is ______________________________ .

1.06 The inverse of $r \rightarrow s$ is ______________________________ .

1.07 The contrapositive of $s \rightarrow t$ is ______________________________ .

1.08 If $p \rightarrow q$ is true and p is true, then we can conclude that q is ________________________ .

Answer true or false (each answer, 1 point),

1.09 __________ Snow is white or grass is red.

1.010 __________ Snow is white and grass is red.

1.011 __________ If snow is white then grass is red.

1.012 __________ If grass is red, then snow is white.

1.013 __________ If snow is not white, then grass is not red.

1.014 __________ If grass is not red, then snow is not white.

1.015 __________ The negation of the negation of the statement: snow is not white.

Write the letter for the equivalent statements (each answer, 2 points).

a. inverse b. contrapositive c. converse d. obtuse

1.016 Conditional and ______________________________ are equivalent.

1.017 a. ______________________________ and b. ______________________________ are equivalent.

Given: If 3 < 2, then -6 < 5. Write the converse, inverse, and contrapositive; *and* indicate which statements are *true* and which are *false* (each answer, 4 points for a., 2 points for b.)

			True or False
1.018 The converse is	a.	______________________________	b. ____________________
1.019 The inverse is	a.	______________________________	b. ____________________
1.020 The contrapositive is	a.	______________________________	b. ____________________

46/58 SCORE ____________ TEACHER ____________ ____________
initials date

Alpha Omega
PUBLICATIONS

804 N. 2nd Ave. E.
Rock Rapids, IA 51246-1759

800-622-3070
www.aop.com

MAT1002 – Apr '15 Printing

ISBN 978-0-86717-632-2

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