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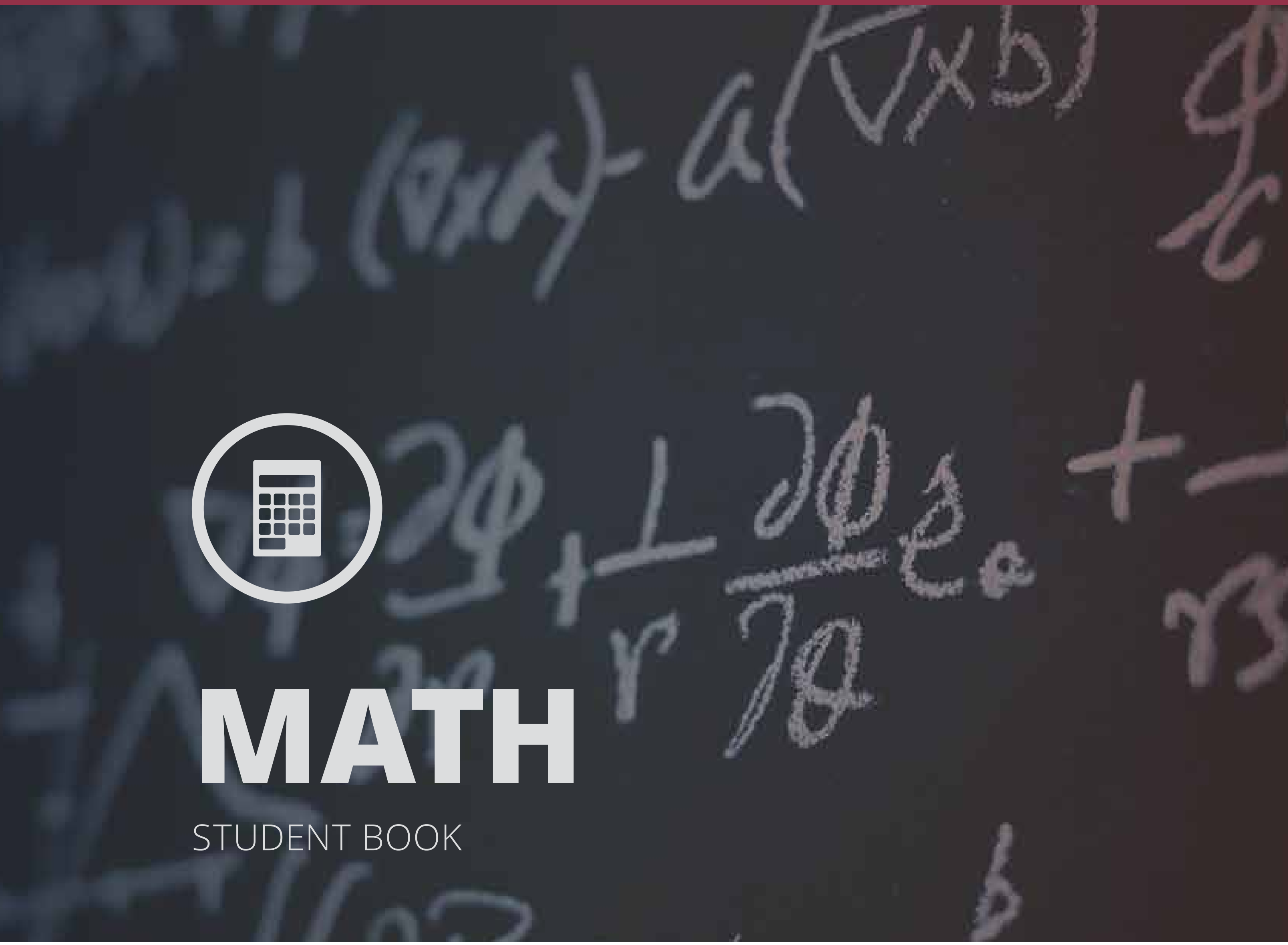

MATH

STUDENT BOOK

12th Grade | Unit 5

Alpha Omega PUBLICATIONS

MATH 1205

ANALYTIC TRIGONOMETRY

LIFEPAC Test is located in the center of the booklet. Please remove before starting the unit.

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Analytic Trigonometry

Introduction

In this unit, algebra is used to study the relationships between the trigonometric functions. Trigonometric expressions are manipulated algebraically to simplify and evaluate them. Equivalent trig expressions and substitution are used in the process of solving trig equations.

Many of the applications connected with the material in this unit are seen in calculus or in sciences that require higher levels of math. Therefore, the unit focuses on logical thinking. Reasoning skills are developed through algebraic proof of trig identities.

Objectives

Read these objectives. The objectives tell you what you will be able to do when you have successfully completed this LIFEPAC®. When you have finished this LIFEPAC, you should be able to:

1. Simplify trigonometric expressions using fundamental trigonometric identities.
2. Determine equivalent trigonometric expressions using fundamental trigonometric identities.
3. Use trig identities to find the remaining trig function values of an angle when one value is known.
4. Solve trig equations using identities and substitution.
5. Determine equivalent trigonometric expressions using the sine, cosine, and tangent addition and subtraction formulas.
6. Evaluate trig functions using the sine, cosine, and tangent addition and subtraction formulas.
7. Evaluate trig functions of half-angle measures.
8. Determine equivalent trigonometric expressions using fundamental trigonometric identities.
9. Express a product of sine and cosine functions as a sum.
10. Express a sum of sine and cosine functions as a product.
11. Determine equivalent trigonometric expressions using product-to-sum and sum-to-product trigonometric identities.

Survey the LIFEPAC. Ask yourself some questions about this study and write your questions here.

1. IDENTITIES AND ADDITION FORMULAS

FUNDAMENTAL TRIGONOMETRIC IDENTITIES

Equations that are true for all values of the variable are called **identities**. You may recall identities, such as $x + 1 = 2x + 2$, from your work in algebra. If an equation containing trig functions is true for all values in the domains of the functions, the equation is called a trigonometric identity.

Trigonometric identities are used to simplify expressions and calculate trig values. They also help you solve equations that would otherwise be impossible to solve.

Trigonometric identities are used extensively in calculus and in fields of study such as sound and optics.

Section Objectives

Review these objectives. When you have completed this section, you should be able to:

- Simplify trigonometric expressions using fundamental trigonometric identities.
- Determine equivalent trigonometric expressions using fundamental trigonometric identities.
- Use trig identities to find the remaining trig function values of an angle when one value is known.

Vocabulary

Study this word to enhance your learning success in this section.

identity . . . . An equation that is true for all values of the variable in its domain.

Note: *All vocabulary words in this LIFEPAC appear in* **boldface** *print the first time they are used. If you are not sure of the meaning when you are reading, study the definitions given.*

RECIPROCAL IDENTITIES

The reciprocal identities are the definitions of the reciprocal functions:

$$\csc\theta = \frac{1}{\sin\theta},\ \sin\theta \neq 0$$

$$\sec\theta = \frac{1}{\cos\theta},\ \cos\theta \neq 0$$

$$\cot\theta = \frac{1}{\tan\theta},\ \tan\theta \neq 0$$

Keep in mind ...

Note that the functions are undefined when the denominator of the fraction is equal to zero. The identities hold true except for these values of θ.

In the following examples, the reciprocal identities are used to simplify expressions.

Example

Simplify $\sin\theta \sec\theta \cos\theta$.

Solution

Use the reciprocal identity to substitute for $\sec\theta$:

$$\sin\theta \sec\theta \cos\theta = \sin\theta\left(\frac{1}{\cos\theta}\right)\cos\theta$$
$$= \sin\theta(1)$$
$$= \sin\theta$$

Example

Simplify $\tan\theta\cos\theta\cot\theta$.

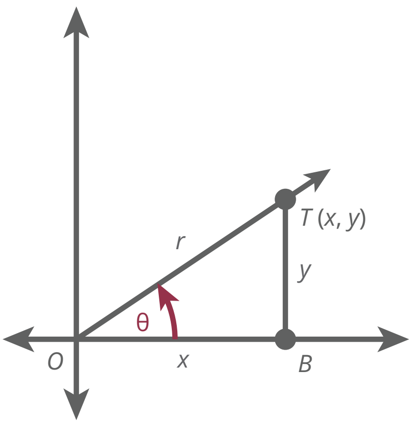
Solution

$$\begin{aligned}\tan\theta\cos\theta\cot\theta &= \tan\theta\cos\theta\left(\frac{1}{\tan\theta}\right)\\ &= \tan\theta\left(\frac{1}{\tan\theta}\right)\cos\theta\\ &= (1)\cos\theta\\ &= \cos\theta\end{aligned}$$

QUOTIENT IDENTITIES

The tangent and cotangent functions can be expressed in terms of sine and cosine.

Consider an angle, θ, in standard position. If a point on the terminal side of θ also lies on the unit circle ($r = 1$), then the coordinates (x, y) represent $(\cos\theta, \sin\theta)$.



Using the fact that $\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$ and substituting for x and y, you get

$$\tan\theta = \frac{\sin\theta}{\cos\theta},\ \cos\theta \neq 0$$

Since the cotangent function is the reciprocal of the tangent function, it can also be written in terms of sine and cosine:

$$\cot\theta = \frac{1}{\tan\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta},\ \sin\theta \neq 0$$

The quotient identities are

$$\tan\theta = \frac{\sin\theta}{\cos\theta},\ \cos\theta \neq 0$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta},\ \sin\theta \neq 0$$

Example

Simplify $\sin\theta\cot\theta$.

Solution

$$\begin{aligned}\sin\theta\cot\theta &= \sin\theta\left(\frac{\cos\theta}{\sin\theta}\right)\\ &= \cos\theta\end{aligned}$$

Example

Simplify

$\cos\theta\sin\theta\csc\theta\sec\theta + \sin\theta\tan\theta\cot\theta$.

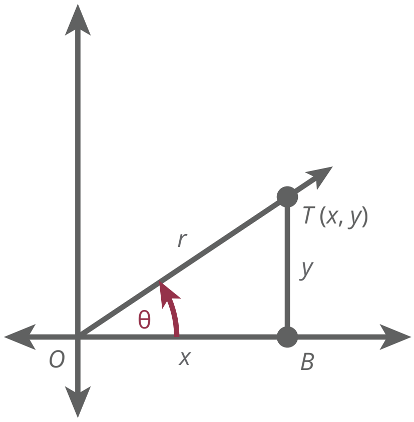
Solution

$\cos\theta\sin\theta\csc\theta\sec\theta + \sin\theta\tan\theta\cot\theta$

$$\begin{aligned}&= \cos\theta\sin\theta\left(\frac{1}{\sin\theta}\right)\left(\frac{1}{\cos\theta}\right) + \sin\theta\tan\theta\left(\frac{1}{\tan\theta}\right)\\ &= \cos\theta\left(\frac{1}{\cos\theta}\right)\sin\theta\left(\frac{1}{\sin\theta}\right) + \sin\theta\tan\theta\left(\frac{1}{\tan\theta}\right)\\ &= 1(1) + \sin\theta\\ &= 1 + \sin\theta\end{aligned}$$

PYTHAGOREAN IDENTITIES

The Pythagorean identities are so named because they are derived from the Pythagorean theorem.



In the triangle shown, if T is on the unit circle, then $x = \cos\theta$, $y = \sin\theta$, and $r = 1$. The Pythagorean theorem gives us

$$x^2 + y^2 = r^2$$

$$\cos^2\theta + \sin^2\theta = 1^2$$

$$\cos^2\theta + \sin^2\theta = 1$$

Two other identities can be derived from the previous identity.

Using the property of equality, divide the equation through by $\cos^2\theta$:

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$1 + \tan^2\theta = \sec^2\theta$$

Using the property of equality, divide the equation through by $\sin^2\theta$:

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\cot^2\theta + 1 = \csc^2\theta$$

Thus, the Pythagorean identities are:

$$\cos^2\theta + \sin^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\cot^2\theta + 1 = \csc^2\theta$$

It is important to realize that algebraic properties of equality allow any identity to be written in other forms. In the following examples, the identities are "algebraically manipulated" to write equivalent expressions.

Examples

1. By subtracting $\cos^2\theta$ from both sides of $\cos^2\theta + \sin^2\theta = 1$, you could write
$$\sin^2\theta = 1 - \cos^2\theta$$
2. Squaring both sides of the identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$ results in
$$\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta}$$
3. If $\cot\theta = \frac{1}{\tan\theta}$, then
$$\tan\theta = \frac{1}{\cot\theta} \text{ for } \cot\theta \neq 0$$

Identities are used to write trigonometric expressions in a simpler form. This might involve reducing the number of trig functions or eliminating a fraction.

Example

Express $\frac{\sin\theta\sec\theta}{\cos^2\theta}$ in terms of $\tan\theta$.

Solution

Look for the trig identities that have tangent in them:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$1 + \tan^2\theta = \sec^2\theta$$

Use substitution to replace the expressions in $\frac{\sin\theta\sec\theta}{\cos^2\theta}$

Write the fraction as a product:

$$\frac{\sin\theta\sec\theta}{\cos^2\theta} = \left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{\sec\theta}{\cos\theta}\right)$$

$$= (\tan\theta)(\sec\theta)\left(\frac{1}{\cos\theta}\right)$$

Use $\sec\theta = \frac{1}{\cos\theta}$ to make a substitution:

$$= (\tan\theta)(\sec\theta)(\sec\theta)$$

$$= \tan\theta\sec^2\theta$$

Use $1 + \tan^2\theta = \sec^2\theta$ to make a substitution:

$$= \tan\theta(1 + \tan^2\theta)$$

$$= \tan\theta + \tan^3\theta$$

Note that the original fraction is undefined when $\cos\theta = 0$. Therefore this identity holds true for all values of θ for which $\cos\theta \neq 0$. So it is true when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

Can you think of an alternate solution for the previous example?

Look at the following alternate solution and examine the differences and similarities in the two solutions.

Example

Express $\frac{\sin\theta\sec\theta}{\cos^2\theta}$ in terms of $\tan\theta$.

Solution

Write the fraction as a product:

$$\frac{\sin\theta\sec\theta}{\cos^2\theta} = (\sin\theta)(\sec\theta)\left(\frac{1}{\cos^2\theta}\right)$$

Use $\sec\theta = \frac{1}{\cos\theta}$ and $\frac{1}{\cos\theta} = \sec^2\theta$ to make substitutions:

$$= (\sin\theta)\left(\frac{1}{\cos^2\theta}\right)(\sec^2\theta)$$

$$= \frac{\sin\theta}{\cos\theta}(\sec^2\theta)$$

Use $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $1 + \tan^2\theta = \sec^2\theta$ to make substitutions:

$$= \tan\theta(1 + \tan^2\theta)$$

$$= \tan\theta + \tan^3\theta$$

In the next example, fractions are added together in order to simplify. Note that the Pythagorean identity is used to make a substitution in the numerator.

Example

$$\frac{\sin^2 \theta}{\cos \theta} + \frac{\cos^2 \theta}{\cos \theta} = \frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta}$$

$$= \frac{1}{\cos \theta}$$

$$= \sec \theta$$

The identities can be used to determine all of the trig function values of an angle when one value is known.

Example

Find the five remaining trig function values of the second-quadrant angle, θ, if $\sec \theta = -\frac{3}{2}$.

Solution

Cosine is the reciprocal of secant.

$$\cos \theta = \frac{1}{\sec \theta} = -\frac{2}{3}$$

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$(-\tfrac{2}{3})^2 + \sin^2 \theta = 1$$

$$\tfrac{4}{9} + \sin^2 \theta = 1$$

$$\sin^2 \theta = \tfrac{5}{9}$$

$$\sin \theta = \pm\frac{\sqrt{5}}{3}$$

$\sin \theta = \frac{\sqrt{5}}{3}$ since the angle is in Quadrant II.

$$\csc \theta = \frac{1}{\sin \theta} = \frac{3\sqrt{5}}{5}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = (\tfrac{\sqrt{5}}{3}) \div (-\tfrac{2}{3}) = -\frac{\sqrt{5}}{2}$$

$$\cot \theta = \frac{1}{\tan \theta} = -\frac{2\sqrt{5}}{5}$$

Reminder:

Reciprocating $\frac{\sqrt{5}}{3}$ results in $\frac{3}{\sqrt{5}}$ which must be rationalized:

$$\frac{3}{\sqrt{5}}\left(\frac{\sqrt{5}}{\sqrt{5}}\right) = \frac{3\sqrt{5}}{5}$$

LET'S REVIEW

Before going on to the practice problems, make sure you understand all the main points of this lesson.

- If an equation containing trig functions is true for all values of the domains of the functions, the equation is called a trigonometric identity.
- Trig identities can be used to simplify trig expressions.
- There may be more than one approach to simplify a trig expression.
- Trig identities may be used to find the remaining trig values of an angle when one value is known.

Multiple-choice questions are presented throughout this unit. To enhance the learning process, students are encouraged to show their work for these problems on a separate sheet of paper. In the case of an incorrect answer, students can compare their work to the answer key to identify the source of error.

Complete the following activities.

1.1 Which of the following statements best describes a trigonometric identity? Select all that apply.

____

a. An equation that holds true for all values of *x*.
b. An equation that holds true for all values of *y*.
c. An equation that holds true for all values of the domain.
d. An equation that holds true for all values of the range.

1.2 Simplify the trigonometric expression $\sec(60)\cos(60)$. _______

a. $\frac{1}{2}$ b. 2 c. 1 d. $\frac{\sqrt{3}}{2}$

1.3 Simplify $\frac{\sin\theta}{\csc\theta}$. _______

a. 1 b. $\cot\theta$ c. $\csc^2\theta$ d. $\sin^2\theta$

1.4 Simplify $(\frac{1}{\csc\theta})(\frac{1}{\sin\theta})$. _______

a. 1 b. $\csc^2\theta$ c. $\sec^2\theta$ d. $\sin^2\theta$

1.5 Simplify $\frac{1}{\cos\theta} + \frac{\tan^2\theta}{\cos\theta}$. _______

a. 1 b. $1 + \sin\theta$ c. $\cos^3\theta$ d. $\sec^3\theta$

1.6 Simplify $\cot\theta\tan^3\theta + 1$. _______

a. 2 b. $\csc^2\theta$ c. $\sec^2\theta$ d. $\tan^2\theta$

1.7 Find $\cot\theta$ if θ terminates in Quadrant III and $\sec\theta = -2$. _______

a. $\pm\sqrt{3}$ b. $\pm\frac{\sqrt{3}}{3}$ c. $\frac{\sqrt{3}}{3}$ d. $\sqrt{3}$

1.8 Simplify $\sqrt{\frac{\tan^2\theta + 1}{\cot^2\theta + 1}}$. _______

a. 1 b. $\cot\theta$ c. $\sec\theta$ d. $\tan\theta$

1.9 Simplify $\csc^2\theta + \cot^2\theta - 1$. _______

a. 0 b. 2 c. $\cot^2\theta$ d. $2\cot^2\theta$

1.10 Simplify $\frac{\cot\theta}{\cos\theta\sec\theta}$. ______

a. 1 b. $\cot\theta$ c. $\cot^2\theta$ d. $\tan\theta$

1.11 Simplify $\sin^2\theta - \sec\theta\cos\theta + \cos^2\theta$. ____________________

1.12 Simplify $\cos\theta(\tan\theta + \cot\theta)$. ______

a. 1 b. $\cos^2\theta$ c. $\csc\theta$ d. $\sec\theta$

Match each trig function with its correct value if θ is an acute angle and $\csc\theta = 2\frac{1}{2}$.

1.13 ____________ $\frac{\sqrt{21}}{5}$

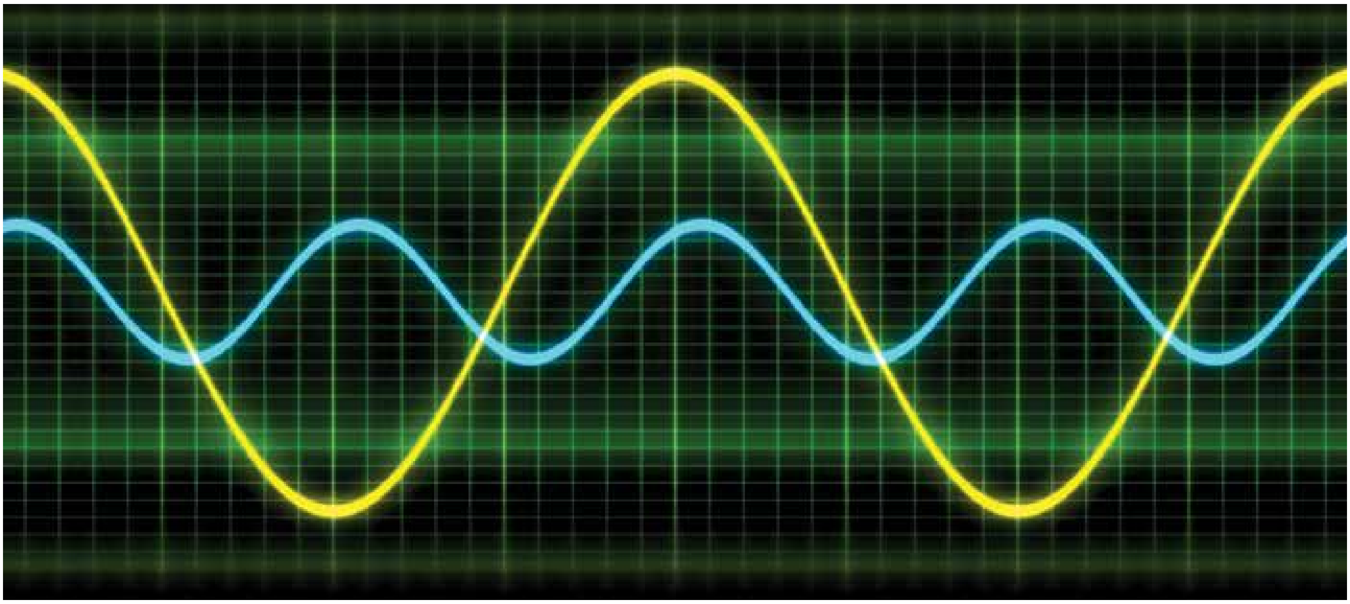
1.14 ____________ $\frac{\sqrt{21}}{2}$

1.15 ____________ $\frac{5\sqrt{21}}{21}$

1.16 ____________ $\frac{2}{5}$

1.17 ____________ $\frac{2\sqrt{21}}{21}$

a. $\tan\theta$
b. $\cot\theta$
c. $\sin\theta$
d. $\cos\theta$
e. $\sec\theta$



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